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ON SOME GENERALIZATION OF THE GOŁĄB–SCHINZEL EQUATION

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Abstract. Inspired by a problem posed by J. Matkowski in [10] we investigate the equation

$$f\big(p(x,y)(xf(y)+y)+(1-p(x,y))(yf(x)+x)\big)=f(x)f(y), \quad x,y\in\mathbb{R},$$

where functions $f\colon \mathbb{R}\to\mathbb{R}$, $p\colon \mathbb{R}^2\to\mathbb{R}$ are assumed to be continuous.

1. Introduction

The composite functional equation

$$f\big(x+yf(x)\big)=f(x)f(y), \quad x,y\in X, \tag{1}$$

where X is a real linear space and $f\colon X\to\mathbb{R}$ is an unknown function, is the well-known Gołąb–Schinzel equation. For details concerning this equation, its origin, generalizations and applications, we refer e.g. to J. Aczél [1], J. Aczél [2, pp. 132-135], J. Aczél, J. Dhombres [3, Chapter 19], J. Aczél, S. Gołąb [4], S. Gołąb, A. Schinzel [5], K. Baron [6], N. Brillouet, J. Dhombres [7], J. Brzdęk [8], P. Javor [9], S. Wołodźko [12].

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There are several papers devoted to some generalizations of equation (1), cf. a survey paper Brzdęk [8], Mureńko [11], J. Matkowski [10]. The last one inspired our paper. In [10] the following generalization of (1) is considered:

$$f\big(p(xf(y)+y)+(1-p)(yf(x)+x)\big)=f(x)f(y),\quad x,y\in X. \tag{2}$$

Roughly speaking, it turns out that the continuous solutions of (2) are the same as the continuous solutions of (1). To be more precise, the main result of J. Matkowski [10] reads as follows:

Theorem 1 ([10]). *Let X be a real linear topological space and $p\in\mathbb{R}$ be fixed. A continuous function $f\colon X\to\mathbb{R}$ satisfies the equation*

$$f\big(p(xf(y)+y)+(1-p)(yf(x)+x)\big)=f(x)f(y),\quad x,y\in X,$$

if, and only if, either

$$f(x)=0,\quad x\in X,$$

or there is an $x^*\in X^*\setminus\{0\}$ such that

$$f(x)=1+x^*(x),\quad x\in X,$$

or $p\in[0,1]$ and there exists $x^*\in X^*\setminus\{0\}$ such that

$$f(x)=\sup(1+x^*(x),0),\quad x\in X.$$

Let a function $f\colon\mathbb{R}\to\mathbb{R}$ be continuous and a function $p\colon\mathbb{R}^2\to\mathbb{R}$ be continuous with respect to each variable. Let $F_{f,p}\colon\mathbb{R}^2\to\mathbb{R}$ be defined by the formula

$$F_{f,p}(x,y)=p(x,y)(xf(y)+y)+(1-p(x,y))(yf(x)+x),\quad x,y\in\mathbb{R}. \tag{3}$$

In this note we consider the generalization of (2) of the form:

$$f(F_{f,p}(x,y))=f(x)f(y),\quad x,y\in\mathbb{R}. \tag{4}$$

The following question naturally arises and was posed in [10]: what are the solutions of equation (4)? Our main result (Theorem 4) states that any real continuous function f fulfilling equation (4) is also of one of the forms described in the Theorem 1.

2. Technical lemmas

For arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ let denote

$$A_c^f = f^{-1}(\{c\})$$

and define $g_f\colon \mathbb{R} \setminus A_1^f \to \mathbb{R}$ by

$$g_f(x) = \frac{x}{1 - f(x)}.$$

2.1. Part I: We establish a form of the function f on the set $f^{-1}((-1,1))$ and a form of the set A_0^f

Lemma 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$, $p\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy equation* (4)*. Then*

(1) $\prod_{j=0}^{n-1}(1 + f(x)^{2^j}) = \frac{1-f(x)^{2^n}}{1-f(x)}, \quad x \notin A_1^f,\ n \in \mathbb{N},$

(2) $f(\prod_{j=0}^{n-1}(1 + f(x)^{2^j})x) = f(x)^{2^n}, \quad x \in \mathbb{R},\ n \in \mathbb{N}.$

Proof. By induction and by using $F_{f,p}(z,z) = z(1 + f(z))$ with

$$z = \prod_{j=0}^{n-1}(1 + f(x)^{2^j})x. \qquad \square$$

Lemma 2. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy equation* (4)*. Then $g_f(f^{-1}((-1,1))) \subseteq A_0^f$.*

Proof. Take arbitrary $x_0 \in f^{-1}((-1,1))$. Then $\lim_{n\to+\infty} f(x_0)^{2^n} = 0$, so Lemma 1 and continuity of f imply that

$$\begin{aligned}
0 = \lim_{n\to+\infty} f(x_0)^{2^n} &= \lim_{n\to+\infty} f\Big(\prod_{j=0}^{n-1}(1 + f(x_0)^{2^j})x_0\Big)\\
&= f\Big(\lim_{n\to+\infty}\prod_{j=0}^{n-1}(1 + f(x_0)^{2^j})x_0\Big)\\
&= f\Big(\lim_{n\to+\infty}\frac{1 - f(x_0)^{2^n}}{1 - f(x_0)}x_0\Big) = f\Big(\frac{x_0}{1 - f(x_0)}\Big).
\end{aligned}$$

Hence $g_f(x_0) \in A_0$. $\square$

Lemma 3. *Let $f\colon \mathbb{R}\to\mathbb{R}$, $p\colon \mathbb{R}^2\to\mathbb{R}$ satisfy equation* (4). *Then $f(0)=0$ or $f(0)=1$*

Proof. Put $x=y=0$ in (4) in order to obtain $f(0)=f(0)^2$. □

Lemma 4. *Let $f\colon \mathbb{R}\to\mathbb{R}$, $p\colon \mathbb{R}^2\to\mathbb{R}$ satisfy equation* (4). *If there exists $x_0\in\mathbb{R}$ such that $f(x_0)=-1$, then $f(0)=1$.*

Proof. Put $x=y=x_0$ in (4) in order to get

$$f(0)=f((1+f(x_0))x_0)=f(x_0)^2=(-1)^2=1. \qquad \square$$

Lemma 5. *Suppose that a continuous function $f\colon \mathbb{R}\to\mathbb{R}$ and a function $p\colon \mathbb{R}^2\to\mathbb{R}$ satisfy equation* (4). *If f is not identically equal zero, then $f(0)=1$.*

Proof. Assume, in search of a contradiction, that f is not identically equal to 0 and $f(0)=0$ (cf. Lemma 3). Let $S_0=(A,B)$ with some $-\infty\le A<0<B\le\infty$ be a component of $f^{-1}((-1,1))$ which contains 0. Then from Lemma 2 it follows that $g_f(S_0)\subseteq A_0^f$ and $0=g_f(0)\in g_f(S_0)$. Moreover, g_f is continuous on $f^{-1}((-1,1))$. So, $g_f(S_0)$ is an interval contained in A_0^f. Since $g_f(0)=0$, we have $g_f(S_0)=|C,D|$ with some $C\le 0\le D$. If $C=0$, then for every $x\in S_0$ we have $g_f(x)=\frac{x}{1-f(x)}\ge 0$, which can occur (in the set $f^{-1}((-1,1))$) only when $x\ge 0$ for every $x\in S_0$, which is impossible since S_0 is open and contains 0. Analogically, $D=0$ can be excluded. Thus $C<0<D$ and at least one of numbers C,D is real (because $f\not\equiv 0$). If for example $D\in\mathbb{R}$ then for every $x\in S_0$ we have $\frac{x}{1-f(x)}\le D$, which is equivalent to $f(x)\le 1-\frac{x}{D}$. Regarding $f(x)\in(-1,1)$ for every $x\in(A,B)$, we conclude that $B\in\mathbb{R}$ and $f(B)=-1$. Then from Lemma 4 we get $f(0)=1$, which contradicts with our assumption. □

Lemma 6. *Suppose that a continuous function $f\colon \mathbb{R}\to\mathbb{R}$ and a function $p\colon \mathbb{R}^2\to\mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *Then set A_0^f is a closed interval or is empty.*

Proof. Assume that $A_0^f\neq\emptyset$. If f is identically equal to 0, then $A_0^f=\mathbb{R}$ and the thesis holds.

If $f\not\equiv 0$, then $f(0)=1$ (cf. Lemma 5). Let $x_0,\ x_1\in A_0^f,\ x_0<x_1$. For every $y\in\mathbb{R}$ we have

$$f(F_{f,p}(x_0,y))=f(x_0)f(y)=0 \quad\text{and}\quad f(F_{f,p}(y,x_1))=f(y)f(x_1)=0,$$

so $F_{f,p}(x_0,\mathbb{R})$ and $F_{f,p}(\mathbb{R},x_1)$ are intervals contained in A_0^f. Obviously,

$$F_{f,p}(0,x_1)=x_1 \quad \text{and} \quad F_{f,p}(x_0,0)=x_0.$$

Furthermore, $F_{f,p}(x_0,x_1)\in F_{f,p}(x_0,\mathbb{R})\cap F_{f,p}(\mathbb{R},x_1)$. Thus,

$$[x_0,x_1]\subseteq F_{f,p}(x_0,\mathbb{R})\cup F_{f,p}(\mathbb{R},x_1)\subseteq A_0^f.$$

Therefore A_0^f is an interval. It is closed, since $A_0^f=f^{-1}(\{0\})$ and the function f is continuous. □

Lemma 7. *Suppose that a continuous function $f\colon\mathbb{R}\to\mathbb{R}$ and a function $p\colon\mathbb{R}^2\to\mathbb{R}$ continuous with respect to each variable satisfy equation (4). If f is not identically equal to 0, then A_0^f is the empty set and $f(\mathbb{R})\subseteq[1,+\infty)$ or there exists $\alpha\in\mathbb{R}^*$ such that either*

(1) $\alpha<0$, $A_0^f=(-\infty,\alpha]$ *and* $f(x)=1-\frac{x}{\alpha}$ *for* $x\in(\alpha,0)$, $f(x)\geq 1$ *for* $x\geq 0$ *or*

(2) $\alpha<0$, $A_0^f=\{\alpha\}$ *and* $f(x)=1-\frac{x}{\alpha}$ *for* $x\in(2\alpha,0)$, $f(x)\leq -1$ *for* $x\leq 2\alpha$, $f(x)\geq 1$ *for* $x\geq 0$ *or*

(3) $\alpha>0$, $A_0^f=[\alpha,+\infty)$ *and* $f(x)=1-\frac{x}{\alpha}$ *for* $x\in(0,\alpha)$, $f(x)\geq 1$ *for* $x\leq 0$ *or*

(4) $\alpha>0$, $A_0^f=\{\alpha\}$ *and* $f(x)=1-\frac{x}{\alpha}$ *for* $x\in(0,2\alpha)$, $f(x)\leq -1$ *for* $x\geq 2\alpha$ *and* $f(x)\geq 1$ *for* $x\leq 0$.

Proof. Assume in search of a contradiction that $A_0^f=[\alpha,\beta]$ with some $-\infty<\alpha<\beta<+\infty$ (cf. Lemma 6).

If $f(x)>0$ for $x>\beta$, $f(x)<0$ for $x<\alpha$ (the case $f(x)<0$ for $x>\beta$, $f(x)>0$ for $x<\alpha$ can be treated similarly), then for $x,y<\alpha$ we have $f(F_{f,p}(x,y))=f(x)f(y)>0$, so $F_{f,p}(x,y)>\beta$. Hence for every $x<\alpha$ we get

$$F_{f,p}(x,\alpha)=\lim_{y\to\alpha^-}F_{f,p}(x,y)\geq\beta$$

and

$$\alpha=F_{f,p}(\alpha,\alpha)=\lim_{x\to\alpha^-}F_{f,p}(x,\alpha)\geq\beta,$$

which is a contradiction with $\alpha<\beta$.

If $f(x)<0$ for $x\in(-\infty,\alpha)\cup(\beta,+\infty)$, then for $x,y<\alpha$, we have $f(F_{f,p}(x,y))=f(x)f(y)>0$, which is impossible.

To finish the proof of the first part of the thesis it is enough to consider the case $f(x)>0$ for $x\in(-\infty,\alpha)\cup(\beta,+\infty)$. Let (γ,δ) be such a component of

$f^{-1}((-1,1))$ that $[\alpha,\beta] \subseteq (\gamma,\delta)$. From Lemma 2 it follows that $g_f((\gamma,\delta)) \subseteq [\alpha,\beta]$. Hence for $x \in (\gamma,\delta)$ we have

$$\alpha f(x) \geq \alpha - x \quad \text{and} \quad \beta f(x) \leq \beta - x.$$

If $\alpha < 0$, then $f(x) \leq 1 - \frac{x}{\alpha}$, so for $x \in (\gamma,\alpha)$ we would have $f(x) < 0$, which contradicts with the assumption.

If $\alpha \geq 0$, then $\beta > 0$ and $f(x) \leq 1 - \frac{x}{\beta}$. Thus, for $x \in (\beta,\delta)$ we would have $f(x) < 0$, which is again a contradiction with the assumption. Therefore either $\alpha = \beta \in \mathbb{R}$ or $\alpha = -\infty$ or $\beta = +\infty$.

If $A_0^f = \emptyset$, then from Lemma 2 it follows that $f^{-1}((-1,1)) = \emptyset$. Lemma 3 and the continuity of f imply $f(\mathbb{R}) \subseteq [1,+\infty)$.

Now assume that $A_0^f \neq \emptyset$ and fix $x_0 \in f^{-1}((-1,1)) \setminus A_0^f$. Then according to Lemma 2 $g_f(x_0) \in A_0^f$. If $A_0^f = \{\alpha\}$, then $g_f(x_0) = \alpha$, so $f(x_0) = 1 - \frac{x_0}{\alpha}$. If $A_0^f = (-\infty,\alpha]$, then $g_f(x_0) \leq \alpha$, so $f(x_0) \geq 1 - \frac{x_0}{\alpha}$ ($\alpha < 0$ because $f(0) = 1$). Hence $f(x_0) = 1 - \frac{x_0}{c}$ with some $c \leq \alpha$. Assume in search of a contradiction that $c < \alpha$. From Lemma 1 it follows that

$$f\Big(x_0 \frac{1 - f(x_0)^{2^n}}{1 - f(x_0)}\Big) = f(x_0)^{2^n}$$

for every $n \in \mathbb{N}$. Thus $f\big(x_0 \frac{1-f(x_0)^{2^n}}{1-f(x_0)}\big) > 0$ for every $n \in \mathbb{N}$. On the other hand,

$$\lim_{n\to+\infty} x_0 \frac{1 - f(x_0)^{2^n}}{1 - f(x_0)} = \frac{x_0}{1 - f(x_0)} = c < \alpha,$$

so there exist $N \in \mathbb{N}$ such that $x_0 \frac{1-f(x_0)^{2^N}}{1-f(x_0)} < \alpha$. Then

$$f\Big(x_0 \frac{1 - f(x_0)^{2^N}}{1 - f(x_0)}\Big) = f(x_0)^{2^N} = 0,$$

which is not possible. To conclude, for every $x_0 \in f^{-1}((-1,1)) \setminus A_0^f$ we have $f(x_0) = 1 - \frac{x_0}{\alpha}$.

Furthermore, if $A_0^f = \{\alpha\}$ and $\alpha < 0$, then for every $x_0 \in f^{-1}((-1,1)) \setminus \{\alpha\}$ we have both $f(x_0) = 1 - \frac{x_0}{\alpha}$ and $f(x_0) \in (-1,1)$, which is possible if and only if $x_0 \in (2\alpha,0)$. Hence $f^{-1}((-1,1)) = (2\alpha,0)$. Moreover, $f(2\alpha) = -1$, $f(0) = 1$, so $f((-\infty,2\alpha]) \subseteq (-\infty,-1]$, $f([0,+\infty) \subseteq [1,+\infty)$. If $A_0^f = \{\alpha\}$ and $\alpha > 0$, then similarly as above we get $f((-\infty,0]) \subseteq [1,+\infty)$ and $f([2\alpha,+\infty) \subseteq (-\infty,-1]$.

Finally, we consider the case of $A_0^f = (-\infty, \alpha]$ with $\alpha < 0$ (the case of $A_0^f = [\alpha, +\infty)$ with $\alpha > 0$ may be analyzed analogically). For every $x_0 \in f^{-1}((-1,1)) \setminus (-\infty, \alpha]$ we have both $f(x_0) = 1 - \frac{x_0}{\alpha}$ and $f(x_0) \in (-1,1)$, which is possible if and only if $x_0 \in (\alpha, 0)$. Hence $f^{-1}((-1,1)) = (-\infty, 0)$ and $f([0,+\infty)) \subseteq [1,+\infty)$. $\square$

2.2. Part II: We prove that if $f \not\equiv 0$, $f \not\equiv 1$ is a solution of (4), then $A_1^f = \{0\}$, so either f takes values greater than 1 for positive arguments and smaller than 1 for negative arguments or the reverse

Lemma 8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $p\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy equation (4). The set A_1^f is a semigroup.*

Proof. Put in (4) $x, y \in A_1^f$ in order to obtain $f(x+y) = 1$. $\square$

Lemma 9. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation (4). If for some $\varepsilon > 0$ we have $f((-\varepsilon, \varepsilon)) \subseteq [1, +\infty)$ or $f((-\varepsilon,\varepsilon)) \subseteq (0,1]$, then $f \equiv 1$.*

Proof. Assume that $f((-\varepsilon, \varepsilon)) \subseteq [1,+\infty)$ for some $\varepsilon > 0$.Observe that $F_{f,p}(0,x) = x = F_{f,p}(x,0)$ for every $x \in \mathbb{R}$. Continuity of $F_{f,p}(\cdot, \varepsilon)$ and $F_{f,p}(\cdot, -\varepsilon)$ at the point 0 implies that there exists $\delta > 0$, $\delta < \varepsilon$ such that for every $|x| < \delta$ we have

$$|F_{f,p}(x,\varepsilon) - \varepsilon| = |F_{f,p}(x,\varepsilon) - F_{f,p}(0,\varepsilon)| < \frac{\varepsilon}{2}$$

and

$$|F_{f,p}(x,-\varepsilon) + \varepsilon| = |F_{f,p}(x,-\varepsilon) - F_{f,p}(0,-\varepsilon)| < \frac{\varepsilon}{2}.$$

Hence

$$F_{f,p}(x,\varepsilon) \in \left(\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}\right) \quad \text{and} \quad F_{f,p}(x,-\varepsilon) \in \left(-\frac{3\varepsilon}{2}, -\frac{\varepsilon}{2}\right), \quad |x| < \delta.$$

For every $|x| < \delta$ from Darboux property of function $F_{f,p}(x,\cdot)$ it follows that there exists $y(x) \in (-\varepsilon, \varepsilon)$ such that $F_{f,p}(x, y(x)) = 0$. Therefore from (4) we have

$$1 = f(0) = f(F_{f,p}(x,y(x))) = f(x)f(y(x)) \geq 1 \quad \text{for } |x| < \delta$$

and equality holds if and only if $f(x) = f(y(x)) = 1$. Thus we have proved that $(-\delta, \delta) \subseteq A_1^f$. However, the set A_1^f is a semigroup (cf. Lemma 8), so $\mathbb{R} = A_1^f$. □

COROLLARY 1. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *If $f^{-1}((-1,1)) = \emptyset$, then $f \equiv 1$.*

PROOF. If $f^{-1}((-1,1)) = \emptyset$, then obviously $A_0^f = \emptyset$, so from Lemma 7 it follows that $f(\mathbb{R}) \subseteq [1, +\infty)$. Therefore, Lemma 9 implies that $f \equiv 1$. □

LEMMA 10. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy equation* (4). *If* 0 *is a leftside accumulation point* (*rightside accumulation point*) *of A_1^f, then $f([0, +\infty)) = \{1\}$ ($f((-\infty, 0]) = \{1\}$).*

PROOF. Let $(x_n)_{n\in\mathbb{N}} \in (A_1^f)^{\mathbb{N}}$ be a decreasing sequence of points tending to 0. Fix $g > 0$. For every $n \in \mathbb{N}$ there exists $l(n) \in \mathbb{N}$ such that $(l(n)-1)x_n < g \le l(n)x_n$. Then $|l(n)x_n - g| < x_n$, so

$$\lim_{n\to+\infty} l(n)x_n = g.$$

Moreover, A_1^f is a semigroup, so $l(n)x_n \in A_1^f$. Thus, A_1^f is dense in $[0, +\infty)$. On the other hand, $A_1^f = f^{-1}(\{1\})$ is closed as a counterimage of a closed set by a continuous function. Hence $f([0, +\infty)) = \{1\}$. □

COROLLARY 2. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *If condition* (1) *or* (2) *from Lemma 7 is satisfied, then there exists $\varepsilon > 0$ such that $f((0, \varepsilon)) \subseteq (1, +\infty)$. If condition* (3) *or* (4) *from Lemma 7 is satisfied, then there exists $\varepsilon > 0$ such that $f((-\varepsilon, 0)) \subseteq (1, +\infty)$.*

PROOF. Assume that condition (1) or (2) from Lemma 7 is fulfilled. From Lemma 7 follows that $f((-\infty, 0)) \subseteq (-\infty, 1)$, $f([0, +\infty)) \subseteq [1, +\infty)$. If the thesis of the corollary did not hold, then 0 would be a righthand side accumulation point of the set A_1^f and Lemma 1 would imply $A_1^f = [0, +\infty)$. Then we would have $f(\mathbb{R}) \subseteq (-\infty, 1]$ and from Lemma 9 we would get $f \equiv 1$, which is a contradiction with the assumption of the lemma.

The proof is similar for condition (3) or (4). □

LEMMA 11. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *If condition* (1) *or* (2) *from Lemma 7 is fulfilled, then $f((0, +\infty)) \subseteq (1, +\infty)$. If condition* (3) *or* (4) *from Lemma 7 is fulfilled, then $f((-\infty, 0)) \subseteq (1, +\infty)$.*

Proof. Without lost of generality we can assume that condition (1) or (2) from Lemma 7 is satisfied.

Assume for contradiction that $(0,+\infty)\cap A_1^f \neq \emptyset$. From Corollary 2 it follows that $\alpha = \inf((0,+\infty)\cap A_1^f) > 0$. Define $h\colon \mathbb{R}\to\mathbb{R}$ by the formula $h(x) = x(1+f(x))$. Then $h([0,\alpha])$ is a compact interval which contains $h(0) = 0$ and $h(\alpha) = 2\alpha$. If there is $\beta \in (\alpha, 2\alpha)$ such that $f(\beta) = 1$, then $\beta = h(\gamma)$ with some $\gamma \in (0,\alpha)$ and according to (4) we would have

$$1 = f(\beta) = f(h(\gamma)) = f(\gamma)^2,$$

which is equivalent to $f(\gamma) = 1$ (cf. Lemma 7). However, this is a contradiction with the definition of α. Thus we proved that $f((\alpha,2\alpha)) \subseteq (1,+\infty)$.

Obviously $h(\alpha) = 2\alpha$, $h(2\alpha) = 4\alpha$, so $[2\alpha, 4\alpha] \subseteq h([\alpha, 2\alpha])$. Hence $3\alpha = h(\gamma)$ with some $\gamma\in(\alpha,2\alpha)$ and $f(3\alpha) = f(h(\gamma)) = f(\gamma)^2 > 1$. On the other hand $3\alpha \in A_1^f$, because A_1^f is a semigroup (cf. Lemma 8). □

2.3. Part III: We establish the form of function f on the set $f^{-1}(\mathbb{R}\setminus(-1,1))$

Theorem 2. *Suppose that a continuous function $f\colon\mathbb{R}\to\mathbb{R}$ and a function $p\colon\mathbb{R}^2\to\mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *If condition* (1) *or* (2) *from Lemma 7 is fulfilled, then $f(x) = 1-\frac{x}{\alpha}$ for $x>0$. If condition* (3) *or* (4) *from Lemma 7 is fulfilled, then $f(x) = 1-\frac{x}{\alpha}$ for $x<0$.*

Proof. Without lost of generality we can assume that condition (1) or (2) from Lemma 7 is satisfied.

Equation (4), Lemma 11 and Lemma 7 imply that for arbitrary $x>0$ there exists exactly one $k(x)\in(\alpha,0)$ such that $f(x)f(k(x)) = 1$. Thus, $f(x) = \frac{\alpha}{\alpha-k(x)}$ for every $x>0$.

Let $x>0$, $\alpha<y<0$. Then $f(x) = \frac{\alpha}{\alpha-k(x)}$, $f(y) = \frac{\alpha-y}{\alpha}$, so $f(x)f(y) = \frac{\alpha-y}{\alpha-k(x)}$. Therefore, from Lemma 7 for $x>0$, $y<0$ we have

$$F_{f,p}(x,y)\in(\alpha,0) \iff f(F_{f,p}(x,y)) = f(x)f(y)\in(0,1) \iff y\in(\alpha,k(x))$$

and

$$F_{f,p}(x,y) > 0 \iff f(F_{f,p}(x,y)) = f(x)f(y) > 1 \iff y\in(k(x),0).$$

Fix $x>0$, $y\in(\alpha,k(x))$. Then

$$f(F_{f,p}(x,y)) = 1-\frac{F_{f,p}(x,y)}{\alpha},$$

$$F_{f,p}(x,y) = p(x,y)\Big(x\Big(1-\frac{y}{\alpha}\Big) + y - y\frac{\alpha}{\alpha - k(x)} - x\Big) + y\frac{\alpha}{\alpha - k(x)} + x$$
$$= p(x,y)\frac{y(xk(x) - \alpha x - \alpha k(x))}{\alpha(\alpha - k(x))} + \frac{\alpha x + \alpha y - xk(x)}{\alpha - k(x)}.$$

Thus

$$f(F_{f,p}(x,y)) = 1 - p(x,y)\frac{y(xk(x) - \alpha x - \alpha k(x))}{\alpha^2(\alpha - k(x))} + \frac{xk(x) - \alpha x - \alpha y}{\alpha(\alpha - k(x))},$$

so

$$1 - p(x,y)\frac{y(xk(x) - \alpha x - \alpha k(x))}{\alpha^2(\alpha - k(x))} + \frac{xk(x) - \alpha x - \alpha y}{\alpha(\alpha - k(x))} = \frac{\alpha - y}{\alpha - k(x)}$$

and

$$\alpha^2(\alpha - k(x)) - p(x,y)y(xk(x) - \alpha x - \alpha k(x)) + \alpha(xk(x) - \alpha x - \alpha y) = \alpha^2(\alpha - y),$$

which implies

$$p(x,y)y(\alpha x + \alpha k(x) - xk(x)) = \alpha(\alpha x + \alpha k(x) - xk(x)).$$

Therefore, either

$$k(x) = \frac{\alpha x}{x - \alpha}, \text{ which is equivalent to } f(x) = 1 - \frac{x}{\alpha},$$

or

$$p(x,y) = \frac{\alpha}{y}.$$

Assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ decreasing to 0 such that $f(x_n) \neq 1 - \frac{x_n}{\alpha}$. Fix $y_0 \in (\alpha, 0)$. Since $\lim_{x\to 0^+} k(x) = 0$, there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have $y_0 \in (\alpha, k(x_n))$, so $p(x_n, y_0) = \frac{\alpha}{y_0}$. Then

$$p(0, y_0) = \lim_{n\to+\infty} p(x_n, y_0) = \lim_{n\to+\infty} \frac{\alpha}{y_0} = \frac{\alpha}{y_0}.$$

Thus,

$$p(0,0) = \lim_{y_0 \to 0^-} p(0, y_0) = +\infty.$$

This contradiction proves that such a sequence $(x_n)_{n\in\mathbb{N}}$ does not exists. So, there is an $\varepsilon > 0$ such that $f(x) = 1 - \frac{x}{\alpha}$ for every $x \in [0, \varepsilon]$.

If $f(x) = 1 - \frac{x}{\alpha}$, $f(y) = 1 - \frac{y}{\alpha}$, then $F_{f,p}(x, y) = x + y - \frac{xy}{\alpha}$ and

$$f(F_{f,p}(x, y)) = f(x)f(y) = 1 - \frac{x + y - \frac{xy}{\alpha}}{\alpha} = 1 - \frac{F_{f,p}(x, y)}{\alpha}.$$

Therefore, $f(z) = 1 - \frac{z}{\alpha}$ for every $z \in F_{f,p}([0, \varepsilon]^2)$. In particular, for $x \in [0, \varepsilon]$ we have $F_{f,p}([0, \varepsilon]^2) \ni F_{f,p}(x, x) = x(1 + f(x)) > 2x$, so $[0, 2\varepsilon] \subseteq F_{f,p}([0, \varepsilon]^2)$ and $f(z) = 1 - \frac{z}{\alpha}$ for every $z \in [0, 2\varepsilon]$. Repeating this reasoning, we get that $f(z) = 1 - \frac{z}{\alpha}$ for every $z > 0$. □

THEOREM 3. *Suppose that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a function $p\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation* (4). *If condition* (2) *from Lemma 7 is fulfilled, then $f(x) = 1 - \frac{x}{\alpha}$ for $x < 0$. If condition* (4) *from Lemma 7 is fulfilled, then $f(x) = 1 - \frac{x}{\alpha}$ for $x > 0$.*

PROOF. Without lost of generality we can assume that condition (2) from Lemma 7 is satisfied.

Suppose that there exist $x < 2\alpha$ such that $f(x) < -1$. Then $x(f(x)+1) > 0$, so from Theorem 2 and (4) we have

$$1 - \frac{x(1 + f(x))}{\alpha} = f(x(1 + f(x))) = f(x)^2,$$

so $\alpha f(x)^2 + xf(x) + x - \alpha = 0$ and solving this quadratic equation we get $f(x) = 1 - \frac{x}{\alpha}$ or $f(x) = -1$. We have chosen x such that $f(x) < -1$, so finally $f(x) = 1 - \frac{x}{\alpha}$.

Let $A = \{x \in (-\infty, 2\alpha)\colon f(x) = -1\}$ and

$$B = \Big\{x \in (-\infty, 2\alpha)\colon f(x) = 1 - \frac{x}{\alpha}\Big\}.$$

The sets A, B are disjoint, their union is $(-\infty, 2\alpha)$ and they are closed in $(-\infty, 2\alpha)$, since the function f is continuous. Connectedness of $(-\infty, 2\alpha)$ implies that $A = \emptyset$ or $B = \emptyset$, so

$$f(x) = -1 \quad \text{for every } x < 2\alpha \quad \text{or} \quad f(x) = 1 - \frac{x}{\alpha} \quad \text{for every } x < 2\alpha.$$

Now we show that the first case leads to a contradiction. Indeed, in this case we would have $f(x) = \max\{-1, 1 - \frac{x}{\alpha}\}$ and we could choose $x_0 > 0, y_0 \leq 2\alpha$ and get $f(F_{f,p}(x_0, y_0)) = f(x_0)f(y_0) = -(1 - \frac{x_0}{\alpha}) < -1$. However, in the considered situation $f(\mathbb{R}) \cap (-\infty, -1) = \emptyset$, which implies the desired contradiction. □

3. Main result

Our main result reads as follows:

THEOREM 4. *Let a continuous function $f\colon \mathbb{R}\to\mathbb{R}$ and a continuous with respect to each variable function $p\colon \mathbb{R}^2\to\mathbb{R}$ satisfy equation (4). Then one of the following conditions is satisfied:*

(1) $f\equiv 0$, *p arbitrary continuous function or*
(2) $f\equiv 1$, *p arbitrary continuous function or*
(3) $f(x)=1-\frac{x}{\alpha}$ *with* $\alpha\neq 0$, *p arbitrary continuous function or*
(4) $f(x)=\max\{0,1-\frac{x}{\alpha}\}$ *with some* $\alpha<0$ *and p being a continuous function satisfying conditions:*
- *if* $x,y\geq\alpha$ *or* $x=y\leq\alpha$ *or* $x=0$ *or* $y=0$, *then* $p(x,y)$ *is arbitrary,*
- *if* $x<y\leq\alpha$, *then* $p(x,y)\leq\frac{\alpha-x}{y-x}$,
- *if* $y<x\leq\alpha$, *then* $p(x,y)\geq\frac{\alpha-x}{y-x}$,
- *if* $x\in(\alpha,0)$, $y<\alpha$, *then* $p(x,y)\geq 1-\frac{\alpha}{x}$,
- *if* $x>0$, $y<\alpha$, *then* $p(x,y)\leq 1-\frac{\alpha}{x}$,
- *if* $x<\alpha$, $y\in(\alpha,0)$, *then* $p(x,y)\leq\frac{\alpha}{y}$,
- *if* $x<\alpha$, $y>0$, *then* $p(x,y)\geq\frac{\alpha}{y}$, *or*

(5) $f(x)=\max\{0,1-\frac{x}{\alpha}\}$ *with some* $\alpha>0$ *and p being a continuous function satisfying conditions:*
- *if* $x,y\leq\alpha$ *or* $x=y\geq\alpha$ *or* $0=y$ *or* $x=0$, *then* $p(x,y)$ *is arbitrary,*
- *if* $x>y\geq\alpha$, *then* $p(x,y)\leq\frac{\alpha-x}{y-x}$,
- *if* $y>x\geq\alpha$, *then* $p(x,y)\geq\frac{\alpha-x}{y-x}$,
- *if* $x<0$, $y>\alpha$, *then* $p(x,y)\leq 1-\frac{\alpha}{x}$,
- *if* $x\in(0,\alpha)$, $y>\alpha$, *then* $p(x,y)\geq 1-\frac{\alpha}{x}$,
- *if* $x>\alpha$, $y\in(0,\alpha)$, *then* $p(x,y)\leq\frac{\alpha}{y}$,
- *if* $x>\alpha$, $y<0$, *then* $p(x,y)\geq\frac{\alpha}{y}$.

Conversely, if functions $f\colon\mathbb{R}\to\mathbb{R}$, $p\colon\mathbb{R}^2\to\mathbb{R}$ *satisfy one of the conditions* $(1)-(5)$, *then* f, p *is a solution of equation* (4).

PROOF. From Lemma 7, Theorem 2, Theorem 3 follows that if f is not identically equal neither to 0 nor to 1, then $f(x)=1-\frac{x}{\alpha}$ or $f(x)=\max\{0,1-\frac{x}{\alpha}\}$. Obviously, if $f(x)=1-\frac{x}{\alpha}$, then the function p is arbitrary. Therefore, to complete the proof it is enough to show that in cases (4) and (5) the function p must satisfy conditions mentioned in, respectively, (4) or (5).

Now we consider the case $f(x)=\max\{0,1-\frac{x}{\alpha}\}$ and $\alpha<0$. For $x,y\geq\alpha$ equation (4) is satisfied independently of $p(x,y)$. For $x,y\leq\alpha$ we have $f(F(x,y))=0$, so $F(x,y)\leq\alpha$ and $F(x,y)=p(x,y)(y-x)+x$. Thus, if $x<y\leq\alpha$, then $p(x,y)\leq\frac{\alpha-x}{y-x}$; if $x=y\leq\alpha$, then $p(x,y)$ is arbitrary; if

$y < x \le \alpha$, then $p(x,y) \ge \frac{\alpha - x}{y - x}$. For $x > \alpha,\ y < \alpha$ we have $f(x)f(y) = 0$, so $F(x,y) \le \alpha$. The definition of F gives

$$\begin{aligned} F(x,y) &= p(x,y)\Big(y - y\Big(1 - \frac{x}{\alpha}\Big) - x\Big) + y\Big(1 - \frac{x}{\alpha}\Big) + x \\ &= -xp(x,y)\Big(1 - \frac{y}{\alpha}\Big) + x + y - \frac{xy}{\alpha} \le \alpha, \end{aligned}$$

so $-xp(x,y)\frac{\alpha - y}{\alpha} \le \frac{1}{\alpha}(\alpha - x)(\alpha - y)$. Thus, $p(0,y)$ are arbitrary; if $x \in (\alpha, 0),\ y < \alpha$, then $p(x,y) \ge 1 - \frac{\alpha}{x}$; if $x > 0,\ y < \alpha$, then $p(x,y) \le 1 - \frac{\alpha}{x}$. Similarly, if $x < \alpha,\ y > \alpha$, then $F(x,y) \le \alpha$ and

$$F(x,y) = p(x,y)\Big(x\Big(1 - \frac{y}{\alpha}\Big) + y - x\Big) + x = yp(x,y)\Big(1 - \frac{x}{\alpha}\Big) + x \le \alpha,$$

so $yp(x,y)\frac{\alpha - x}{\alpha} \le \alpha - x$. Thus, $p(x,0)$ are arbitrary; if $x < \alpha, y \in (\alpha, 0)$, then $p(x,y) \le \frac{\alpha}{y}$; if $x < \alpha, y > 0$, then $p(x,y) \ge \frac{\alpha}{y}$.

The case (5) is treated analogically to the case (4).

It is easy to check that function fulfilling one of the conditions (1)–(5) is a solution of (4). $\square$

In the end observe, that there exist a lot of continuous functions p which satisfy conditions from (4) or (5) of Theorem 4, e.g. for $\alpha > 0$ one may take

$$p_0(x,y) = \begin{cases} \frac{\alpha}{y}, & \text{for } |y| \ge \frac{\alpha}{2} \\ \frac{4y}{\alpha}, & \text{for } |y| < \frac{\alpha}{2}. \end{cases}$$

Let $p_1 \colon \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function continuous with respect to each variable and such that $p_1(x,y) \neq 0$ only for $x < 0$ and $y < 0$. Then the function $p_0 + p_1$ satisfies conditions (5) of Theorem 4, too.

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